数理解析研究所講究録 1452 巻 2005 年 22-29

ULT-minimal realization of piecewise linear functions *

櫻井 智章, 室伏 俊明
Tomoaki Sakurai, Toshiaki Murofushi
東京工業大学大学院 総合理工学研究科 知能システム科学専攻
Departnent of Computationanl Intelligence and Systems Science,
Tokyo Institute of Technology

Abstract: A correspondence f is called a linear complementarity correspondence if it has a state-variable representation, which can be reformulated as the linear complementarity problem, in other words, calculating function values results in solving the associated linear complementarity problem. The paper formulates the minimum-dimension state-variable representation problem, called the minimal realization problem, and discusses the criterion for a given representation to be a minimal realization. Forthermore, in this argument, the new concepts concerning redundancy in the state-variables will be given.

1 Introduction

Various studies on piecewise linear functions are known in the literature, and presented mainly from practical point of view [6] [8]. In due course, however, the importance of a representation for piecewise linear functions becomes widely recognized. In this paper, we characterize a piecewise linear function as a continuous and linear function on each polyhedron of some finite family obtained by domain-partitioning (cf. [4] [9]). van Bokhoven et al. [8] introduced a state-variable representation to model non-linear electric/electronic circuits as a piecewise linear model. It has been shown that every piecewise linear function can be expressed in such a representation [5]. There are various analyses and modelling techniques done by using this representation [3] [8] since their pioneering work. As we discuss briefly in Section 2 many questions are left unanswered at the current stage of understanding of the state-variable representation. In particular, we are intrigned by the fact that the state-variable representation is not unique, and that infinitely many choices of the dimension of state-variables are possible. The objective of this paper is to explain a method of finding a minimum-dimension state-variable representation, which we dub a minimal realization problem, for every piecewise linear function.

The paper is constructed as follows: In Section 2, we explain the state-variable representation, and propose the questions concerning the minimal realization problem. In Section 3, we formulate the minimal realization problem, and report our investigation on the ULT-representation, the notion of which will be introduced in Section 2. We will then propose criteria for a particular representation to be a minimal realization. In Section 4, the conclusion will be briefly discussed.

Throughout this paper, m and n indicate positive integer. A^T denotes the transposition of a matrix (or a vector) A. The max operator is denoted by $\vee$, and for $x \in \mathbb{R}$ we write $x^+ = x \vee 0$. The inner product of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. Unless otherwise noted, k is a nonnegative integer. "Linear" should be read as "affine linear" in this paper.

*This work is partially supported by a grant from the Ministry of Education, Culture, Sports, Science and Technology, the 21st Century COE Program "Creation of Agent-Based Social Systems Sciences."

2 State-variable representation

Definition 1. (See [8]) The correspondence f from $\boldsymbol{x} \in \mathbb{R}^n$ to $\boldsymbol{y} \in \mathbb{R}^m$ is called a *linear complementarity correspondence*, an LCC for short, if there exist a nonnegative integer k and matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$, $C \in \mathbb{R}^{k \times n}$, $D \in \mathbb{R}^{k \times k}$, and vectors $\boldsymbol{g} \in \mathbb{R}^m$, $\boldsymbol{h} \in \mathbb{R}^k$ such that

$$\boldsymbol{y} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{g}, \tag{1}$$

$$\boldsymbol{j} = C\boldsymbol{x} + D\boldsymbol{u} + \boldsymbol{h}, \tag{2}$$

$$\boldsymbol{u}, \boldsymbol{j} \geqq \boldsymbol{0}, \ \langle \boldsymbol{u}, \boldsymbol{j} \rangle = 0. \tag{3}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{j}$ are called *state-variables*, and the equations (1)–(3) are collectively called a *state variable representation*. We abbreviate the above representation as $(\mathcal{A}, \mathcal{C})$ for $\mathcal{A} = (A, B, \boldsymbol{g})$ and $\mathcal{C} = (C, D, \boldsymbol{h})$. By convention a state-variable representation with $k = 0$ will be denoted by $(A, \boldsymbol{g})$.

Remark 1. Every linear function is an LCC having a representation $(A, \boldsymbol{g})$.

In the state-variable representation, the problem of finding $\boldsymbol{y}$ for each $\boldsymbol{x}$ is reduced to a linear complementarity problem (an LCP for short) by substituting $\boldsymbol{q}(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{h}$; that is, in order to calculate a function value, we must solve the LCP $(D, \boldsymbol{q}(\boldsymbol{x}))$ each $\boldsymbol{x}$. See the Appendix for the definition of linear complementarity problem. Together with the NP-completeness of the LCP this makes it computationally demanding to calculate correspondence values. van Bokhoven et al. have proposed a method of transforming a state-variable representation into an "explicit representation" with respect to $\boldsymbol{x}$, and overcome this difficulty. So far, the method is known to be applicable when a P- or ULT-representation is considered, which are described in Definition 2 [1] [7]. We will not discuss their method in detail here since it is not our main concern. But we note that the method has a substantial role in proving Theorem 1 of this section.

Here, we will define the P- and ULT-representations. See the Appendix for the definition of P- and ULT-matrices.

Definition 2. (cf. [1] [8]) (a) A state-variable representation is called a *P-representation* if the matrix D in (2) is a P-matrix. The family of LCC's having a P-representation is called *Class P*, and denoted by $\mathcal{P}$.
(b) A state-variable representation is called a *ULT-representation* if the matrix D in (2) is a ULT-matrix. The family of LCC's having a ULT-representation is called *Class ULT*, and denoted by $\mathcal{ULT}$.

Remark 2. By convention we will assume that $(A, \boldsymbol{g})$ is both a P- and ULT-representation. Thus, every linear function belongs to both $\mathcal{P}$ and $\mathcal{ULT}$. In general, an LCC is a multi-valued function. But, by Proposition A.1 in Appendix, an LCC in Class P becomes a simgle-valued function. It is clear by defintion that $\mathcal{P} \subset \mathcal{ULT}$. In fact, Classes P and ULT coincide, that is, $\mathcal{P} = \mathcal{ULT}$, as we will explain below.

The following Theorem 1 shows that Classes P and ULT are subclasses of the family of all piecewise linear functions ($\mathcal{PWL}$ for short).

Theorem 1. [1] [7] *Every LCC having a P- or ULT-representation is a piecewise linear function.*

Moreover, the following theorem has been established [10].

Theorem 2. *Every piecewise linear function has a ULT-representation.*

Combining Theorem 1 and Theorem 2 together, we can claim

Colorrarly 1. $\mathcal{P} = \mathcal{ULT} = \mathcal{PWL}$.

Next, we will introduce the notion of the "derived" LCP, which is used in the rest of the paper.

Definition 3. Let k be a positive integer, and let $C \in \mathbb{R}^{k\times n}$, $D \in \mathbb{R}^{k\times k}$, $\boldsymbol{h} \in \mathbb{R}^k$ be given. For each $\boldsymbol{x} \in \mathbb{R}^n$, we define the LCP $(D, \boldsymbol{q}(\boldsymbol{x}))$, where $\boldsymbol{q}(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{h}$. We call this the *derived LCP from* $(C, D, \boldsymbol{h})$.

In the rest of the section, we assume that D is a P-matrix. Then the solution $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{j}(\boldsymbol{x})$ to $(D, \boldsymbol{q}(\boldsymbol{x}))$ is uniquely determined for each $\boldsymbol{x} \in \mathbb{R}^n$. Thus, the correspondence $\boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{x} \mapsto \boldsymbol{j}(\boldsymbol{x})$ are single-valued functions. In addition, it has been shown that they are piecewise linear functions [11].

The following Lemma 1 is an immediate concequence of the fact that a solution to $(D, \boldsymbol{q}(\boldsymbol{x}))$ is nonnegative. It is used later in the proof of Theorem 3.

Lemma 1. *Let $(D, \boldsymbol{q}(\boldsymbol{x}))$ be the derived LCP from $(C, D, \boldsymbol{h})$ with a P-matrix D, and let $\boldsymbol{u}(\boldsymbol{x}) = (u_p(\boldsymbol{x}))_{p=1}^k$ be the unique solution to $(D, \boldsymbol{q}(\boldsymbol{x}))$. Then, for each $p = 1, 2, \ldots, k$, $u_p(\boldsymbol{x})$ is a linear function of $\boldsymbol{x}$ if and only if $u_p(\boldsymbol{x})$ is constant on $\mathbb{R}^n$.*

Every LCC has possibly many different state-variable representations. In fact, if an LCC has a state-variable representation for some k-dimensional state-variables, then it has a k'-dimensional representation for every $k' > k$. This leads us to the following questions: (i) What is the minimum dimension of state-variables ? (ii) How does one find a minimum-dimension representation to every LCC ? We will report our investigation on these questions in the following Section 3.

3 ULT-minimal realization

In this section, we shall reformulate two questions raised in the end of Section 2 more or less in a regorous manner, and introduce a minimal realization problem. In particular, we will cast it into the ULT-representation, and propose several critaria for the sufficiency of minimal realization.

3.1 Minimal realization problem

In this subsection, we will present the notion of minimal realization problem. Before introducing the notion, we begin with several key notations. Here we will make it implicit that the LCC f of interest has the range in m-dimensional space, and has the domain in n-dimensional space. For positive integer k, we define the classes $\mathbb{A}^k$ and $\mathbb{C}^k$ respectively by:

$$\mathbb{A}^k = \{(A, B, \boldsymbol{g}) \mid A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times k},\ \boldsymbol{g} \in \mathbb{R}^m\},$$
$$\mathbb{C}^k = \{(C, D, \boldsymbol{h}) \mid C \in \mathbb{R}^{k\times n},\ D \in \mathbb{R}^{k\times k},\ \boldsymbol{h} \in \mathbb{R}^k\}.$$

Then we can define the class of all f's having a representation with k-dimensional state-variables by $\mathbb{S}^k = \mathbb{A}^k \times \mathbb{C}^k$. For the sake of convenience we set $\mathbb{S}^0 = \{(A, \boldsymbol{g}) \mid A \in \mathbb{R}^{m\times n},\ \boldsymbol{g} \in \mathbb{R}^m\}$ to express the class of all the linear functions having m-dimensional range space. Then $\mathbb{S} = \bigcup_{k\geq 0} \mathbb{S}^k$ gives the class of all state-variable representations. Moreover, by $\mathbb{S}^k_{\mathrm{ULT}}$ we denote the class $\mathbb{S}^k$ with D's restricted to ULT-matrices. Then $\mathbb{S}_{\mathrm{ULT}} = \bigcup_{k\geq 0} \mathbb{S}^k_{\mathrm{ULT}}$ similarly gives the class of all ULT-representations. Note that $\mathbb{S}^0_{\mathrm{ULT}} = \mathbb{S}^0$. Finally we define the subclass $\mathbb{S}(f)$ of $\mathbb{S}$ which represents a particular f. In the same manner, by $\mathbb{S}_{\mathrm{ULT}}(f)$ we denote the class of all ULT-representations of f.

Now let us explain the notion of minimal realization problem. First, we will introduce the "realization dimension" of a representation in Definition 4.

Definition 4. (i) Let $\mathcal{S} \in \mathbb{S}$, and let k be a nonnegative integer. We call k the *realization dimension* of $\mathcal{S}$ if $\mathcal{S} \in \mathbb{S}^k$, denoted by $\dim(\mathcal{S})$.
(ii) Let f be an LCC, and let $\mathcal{S} \in \mathbb{S}(f)$. Then $\mathcal{S}$ is called a *minimal realization* of f if $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$ for every $\mathcal{T} \in \mathbb{S}(f)$.

Then the minimal realization problem will be described as follows.

Problem 1. The *minimal realization problem* associated with an LCC f will be referred to the following two problems:
(a) Decide whether or not a particular $\mathcal{S} \in \mathbb{S}(f)$ is a minimal realization of f;
(b) find a minimal realization of f if the above candidate $\mathcal{S}$ is not a minimal realization.

Since it will be difficult to solve a "general" problem, we will restrict our investigation to the ULT-representation. In Definition 5, we give a ULT-minimal realization in the same manner to Definition 4.

Definition 5. A representation $\mathcal{S} \in \mathbb{S}_{\mathrm{ULT}}(f)$ is called a *ULT-minimal realization* of f if $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$ for every $\mathcal{T} \in \mathbb{S}_{\mathrm{ULT}}(f)$.

Then the ULT-minimal realization problem will be defined as follows.

Problem 2. The *ULT-minimal realization problem* associated with an LCC f will be referred to the following two problems:
(a) Decide whether or not a particular $\mathcal{S} \in \mathbb{S}_{\mathrm{ULT}}(f)$ is a ULT-minimal-realization of f;
(b) find a ULT-minimal realization of f if the above candidate $\mathcal{S}$ is not a ULT-minimal realization of f.

In the following Subsection 3.2, we will discuss a criterion for a particular ULT-representation to be a ULT-minimal realization. Since we discuss only ULT-representation for the rest of this paper, we simply write "representation" and "minimal realization" for "ULT-representation" and "ULT-minimal realization", respectively.

The following relation $\cong$ on $\mathbb{S}$ will be used in Subsection 3.2.

Definition 6. Two state-variable representations $\mathcal{S}, \mathcal{T} \in \mathbb{S}$ are said to be *equivalent* to each other, denoted by $\mathcal{S} \cong \mathcal{T}$, if there exists an LCC f such that $\mathcal{S}, \mathcal{T} \in \mathbb{S}(f)$.

3.2 ULT-irreducibility

In the previous subsection, we have formulated the ULT-minimal realization problem. In solving the problem, one hopes to understand how to determine whether or not a given representation is a minimal realization. In this subsection, we will discuss the criteria for a representation to be a minimal realization.

Before we turn to the actual criteria, it will be beneficial to discuss the following three examples. These examples demonstrate that if the state-variable representation has redundancy in a certain sense then such representation is not a minimal realization.

Example 1. Let $\mathcal{S}_1 = (\mathcal{A}_1, \mathcal{C}_1)$ be a ULT-representation given by the following:

$$A_1 = \begin{pmatrix} 1 & 1 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix},\ C_1 = \begin{pmatrix} -3 & -6 \\ -4 & -8 \\ 4 & 8 \\ 6 & 12 \end{pmatrix},\ D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ g_1 = 0,\ \boldsymbol{h}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we can establish the following relations among the state-variables $u_i(\boldsymbol{x})$ $(i = 1, 2, 3, 4)$:

$$u_1(\boldsymbol{x}) = 3u_2(\boldsymbol{x}),\ u_3(\boldsymbol{x}) = 2u_4(\boldsymbol{x}),\ u_4(\boldsymbol{x}) = -2x_1 - 4x_2 + 2u_2(\boldsymbol{x}).$$

Thus, the variables $u_1(\boldsymbol{x})$, $u_3(\boldsymbol{x})$, $u_4(\boldsymbol{x})$ can be eliminated from $\mathcal{S}_1$, and therfore, $\mathcal{S}_1$ becomes equivarent to the following ULT-representation $\mathcal{S}_1'$:

$$A_1' = \begin{pmatrix}-1 & -3\end{pmatrix},\ B_1' = 3,\ C_1' = \begin{pmatrix}1 & 2\end{pmatrix},\ D_1' = 1,\ g_1' = 0,\ h_1' = 0.$$

In the above sense, we say that the variables $u_1(\boldsymbol{x})$, $u_3(\boldsymbol{x})$, $u_4(\boldsymbol{x})$ are redundant.

Example 2. Let $\mathcal{S}_2 = (\mathcal{A}_2, \mathcal{C}_2)$ be a ULT-representation given by the following:

$$A_2 = \begin{pmatrix}1 & 1\end{pmatrix},\ B_2 = \begin{pmatrix}0 & 1\end{pmatrix},\ C_2 = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\ D_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ g_2 = 0,\ \boldsymbol{h}_2 = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Then the state-variables $u_1(\boldsymbol{x})$ and $u_2(\boldsymbol{x})$ in this example do not have the same redundancy as in Example 1. However, since the first column of B_2 is equal to 0 and the function $u_2(\boldsymbol{x})$ is independently calculated from $u_1(\boldsymbol{x})$, the variable $u_1(\boldsymbol{x})$ is unnecessary. Thus $\mathcal{S}_2$ becomes equivarent to the following lower dimensional ULT-representation $\mathcal{S}_2'$:

$$A_2' = \begin{pmatrix}1 & 1\end{pmatrix},\ B_2' = 1,\ C_2' = \begin{pmatrix}2 & 1\end{pmatrix},\ D_2' = 1,\ g_2' = 0,\ h_2' = 0.$$

In the above sense, we say that $u_1(\boldsymbol{x})$ is redundant.

Example 3. Let $\mathcal{S}_3 = (\mathcal{A}_3, \mathcal{C}_3) \in \mathbb{S}^k_{\mathrm{ULT}}$ be given. Suppose there are a positive integer $k' < k$, $\mathcal{C}_3' \in \mathbb{C}^{k'}_{\mathrm{ULT}}$ and a matrix $E \in \mathbb{R}^{k \times k'}$ such that the solution $\boldsymbol{u}(\boldsymbol{x})$ to the derived LCP from $\mathcal{C}_3$ can be expressed as $\boldsymbol{u}(\boldsymbol{x}) = E\boldsymbol{u}'(\boldsymbol{x})$, where $\boldsymbol{u}'(\boldsymbol{x})$ is the solution to the derived LCP from $\mathcal{C}_3'$. Then $\mathcal{S}_3$ becomes equivalent to the ULT-representation $\mathcal{S}_3' = (\mathcal{A}_3', \mathcal{C}_3') \in \mathbb{S}^{k'}_{\mathrm{ULT}}$, where $\mathcal{A}_3' = (A_3, B_3E, \boldsymbol{g}_3)$ for $\mathcal{A}_3 = (A_3, B_3, \boldsymbol{g}_3)$. In the above sense, we say that the state-variables $\boldsymbol{u}(\boldsymbol{x})$ contain redundancy.

We will now discuss the criteria in detail. As demonstrated above, if the state-variables of a representation have redundancy, then the representation is not a minimal realization. Moreover, it turn out that there exist the following causes of redundancy at least: (i) the existence of dependence of the state-variables (Ex.1), (ii) the existence of unnecessary state-variables resulting from a column of B becoming 0 (Ex.2), (iii) the existence of another low dimensional state-variables restoring original state-variables (Ex.3). Conversely, if a given representation is not a minimal realization, then the state-variables should have certain redundancy.

From the above argument, we expect that a minimal realization is characterized by the question on whether or not there is redundancy in state-variables. Then, what kind of redundancy should be investigated? This question has remained unsettled.

So far, we have investigated ULT-reducibility generalizing the redundancy of (iii), and found that the redundancy of (i) and ULT-reducibility are equivalent. Here we will give a full account of this investigation.

First, we will give a notion of ULT-reducibility generalizing the redundancy of (iii).

Definition 7. Let $\mathcal{C} \in \mathbb{C}^k_{\mathrm{ULT}}$. Then $\mathcal{C}$ is said to be *ULT-reducible* if there exists some $\mathcal{C}' \in \mathbb{C}^{k'}_{\mathrm{ULT}}$ with $k' < k$ such that every $\mathcal{A} \in \mathbb{A}^k$ of arbitrary dimension m of range space has a reduced representation $(\mathcal{A}', \mathcal{C}')$ equivalent to $(\mathcal{A}, \mathcal{C})$ [i.e., $(\mathcal{A}, \mathcal{C}) \cong (\mathcal{A}', \mathcal{C}')$]. Here we include the special case $k' = 0$ in which $(\mathcal{A}, \mathcal{C})$ can be found representing a linear function. If not ULT-reducible, it is said to be *ULT-irreducible*.

$\mathcal{C}_3$ in Example 3 is ULT-reducible. Moreover, by Theorem 3 below, $\mathcal{C}_1$ in Example 1 is also ULT-reducible. On the other hand, $\mathcal{C}_2$ in Example 2 is ULT-irreducible.

The following Propsition 1 is an immediate concequence of Definition 5 and Definition 7. Propsition 1 shows that ULT-irreducibility of $\mathcal{C}$ is necessary for ULT-minimal realization. Note that Example 2 is a counterexample for the sufficiency.

Proposition 1. *If $\mathcal{S} = (\mathcal{A}, \mathcal{C}) \in \mathbb{S}_{\mathrm{ULT}}(f)$ is a ULT-minimal realization for f, then $\mathcal{C}$ is ULT-irreducible.*

The following Theorem 3 shows that the redundancy of (i) and ULT-reducibility of $\mathcal{C}$ is equivalent. The condition (S) in Theorem 3 characterizes a certain kind of dependency among the components of the state-variables.

Theorem 3. *Let k be a positive integer. Then $\mathcal{C} \in \mathbb{C}^k_{\mathrm{ULT}}$ is ULT-reducible if and only if the solution $\boldsymbol{u}(\boldsymbol{x})$ to the derived LCP from $\mathcal{C}$ satisfies the following condition (S):*
(S) For some $p = 1, 2, \ldots, k$, there exist $\{\lambda_i\}_{i<p} \subset \mathbb{R}$ and a linear function $l_p : \mathbb{R}^n \to \mathbb{R}$ such that

$$u_p(\boldsymbol{x}) = \sum_{i<p} \lambda_i u_i(\boldsymbol{x}) + l_p(\boldsymbol{x}) \quad (\forall \boldsymbol{x} \in \mathbb{R}^n).$$

Outline of the proof. Let $\mathcal{C} \in \mathbb{C}^k_{\mathrm{ULT}}$, and let $\boldsymbol{u}(\boldsymbol{x})$ be the solution to its derived LCP.

Sufficiency: If $p = 1$, then $u_1(\boldsymbol{x})$ is linear, hence a constant $a \geqq 0$ by Lemma 1. Set $\mathcal{C}' = (C', D', \boldsymbol{h}') \in \mathbb{C}^{k'}_{\mathrm{ULT}}$, where $k' = k - 1$, $C' = (\boldsymbol{c}_2^T, \ldots, \boldsymbol{c}_k^T)^T$, $D' = (d_{i,j})_{i,j \neq 1}$ and $\boldsymbol{h}' = (h_{i+1} + a d_{i+1,1})_{i=1}^{k'}$ for $C = (\boldsymbol{c}_1^T, \ldots, \boldsymbol{c}_k^T)^T$, $D = (d_{i,j})_{1 \leqq i,j \leqq k}$ and $\boldsymbol{h} = (h_i)_{i=1}^k$. For any $\mathcal{A} \in \mathbb{A}^k$, we can find $\mathcal{A}' \in \mathbb{A}^{k'}$ such that $(\mathcal{A}, \mathcal{C}) \cong (\mathcal{A}', \mathcal{C}')$. In a similar manner, we can construct such $\mathcal{C}'$, in the case of $p > 1$. Therefore, $\mathcal{C}$ is ULT-reducible.

Necessity: Suppose $\mathcal{C}$ is ULT-reducible. Choose the dimension of range space as $m = k$. Then there exist a number $k' < k$, $\mathcal{C}' \in \mathbb{C}^{k'}_{\mathrm{ULT}}$ and $\mathcal{A}' = (A', B', \boldsymbol{g}') \in \mathbb{A}^{k'}$ such that

$$\boldsymbol{u}(\boldsymbol{x}) = B' \boldsymbol{u}'(\boldsymbol{x}) + l(\boldsymbol{x}) \quad (\forall \boldsymbol{x} \in \mathbb{R}^n),$$

where $l(\boldsymbol{x}) = A'\boldsymbol{x} + \boldsymbol{g}'$, and $\boldsymbol{u}'(\boldsymbol{x})$ is the solution to the derived LCP from $\mathcal{C}'$. Since $k' < k$, we have a nonzero vector $\boldsymbol{\lambda} \in \mathbb{R}^k$ such that $(B')^T \boldsymbol{\lambda} = \boldsymbol{0}$, and hence $\boldsymbol{\lambda}^T \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{\lambda}^T l(\boldsymbol{x})$. This implies that $\boldsymbol{u}(\boldsymbol{x})$ satisfies the condition (S). □

4 Concluding remarks

This paper has introduced the ULT-minimal realization problem associated with the state-variable representation, and discussed the relation between the criteria for ULT-minimal realization and redundancy in the state-variables. As a result of this investigation, we have proposed the concept of ULT-reducibility, which is one of the necessary conditions for ULT-minimal realization. Moreover, it has been shown that this characterizes the certain kinds of dependency among the state-variables. However, there has been no efficient algorithms to check such a condition. A further study will be needed to establish such an algorithm. Moreover, the question of what kinds of the criteria will give the complete characterization for ULT-minimal realization remains to be open. Such a characterization is to be investegated.

A Appendix: The linear complementarity problem

Let k be a positive integer.

Definition A.1. [2] Given a matrix $D \in \mathbb{R}^{k\times k}$ and a vector $\boldsymbol{q} \in \mathbb{R}^k$, a *linear complementarity problem*, LCP for short, is to find a pair of vectors $\boldsymbol{u}, \boldsymbol{j} \in \mathbb{R}^k$ such that

$$\boldsymbol{j} = D\boldsymbol{u} + \boldsymbol{q}, \tag{4}$$

$$\boldsymbol{u}, \boldsymbol{j} \geq \boldsymbol{0}, \ \langle \boldsymbol{u}, \boldsymbol{j} \rangle = 0 \tag{5}$$

or to show that no such pair exists. We denote the above problem by the pair $(D, \boldsymbol{q})$. A pair $(\boldsymbol{u}, \boldsymbol{j})$ satisfying (5) is called complementary, and the one satisfying (4) and (5) is called a solution to the LCP $(D, \boldsymbol{q})$.

Next, we will introduce the two matrices, in relation with a representation of piecewise linear function.

Definition A.2. A square matrix is said to be:
(i) [2] *P-matrix* if all its principal minors are positive;
(ii) [7] *unit lower triangular matrix*, ULT-matrix for short, if it is a lower trianguler matrix and its diagonal elements are all 1's.

Remark 3. A principal minor is the determinant of a principal submatrix of D, and a principal submatrix is formed by deleting exactly the same members of rows and columns from the original matrix. It is easy to see that every ULT-matrix is a P-matrix.

In general, the LCP does not necessarily have a solution. Even if it has a solution, generally it is not necessarily unique. However, Proposition A.1 below claims that a P-matrix guarantees the uniqueness of solution.

Proposition A.1. [2] *A matrix $D \in \mathbb{R}^{k\times k}$ is a P-matrix if and only if the LCP $(D, \boldsymbol{q})$ has a unique solution for every $\boldsymbol{q} \in \mathbb{R}^k$.*

By Proposition A.1, if D is a P-matrix, then the pair of vectors $\boldsymbol{u}$ and $\boldsymbol{j}$ satisfying (4) and (5) is uniquely determined. In such a case, we often refer $\boldsymbol{u}$ (or $\boldsymbol{j}$) as "the unique solution to $(D, \boldsymbol{q})$", without confusion.

Remark 4. When the matrix D is nonsingular, we can define the LCP $(D^{-1}, -D^{-1}\boldsymbol{q})$ for each $\boldsymbol{q} \in \mathbb{R}^k$. Then $(\boldsymbol{u}, \boldsymbol{j})$ is a solution to $(D, \boldsymbol{q})$ if and only if $(\boldsymbol{j}, \boldsymbol{u})$ is a solution to $(D^{-1}, -D^{-1}\boldsymbol{q})$. Moreover, by Proposition A.1, if D is a P-matrix, then the unique solution to $(D, \boldsymbol{q})$ is also the unique solution to $(D^{-1}, -D^{-1}\boldsymbol{q})$. Thus, we obrain

Proposition A.2. *The inverse of a P-matrix is also a P-matrix.*

References

[1] W. M. G. van Bokhoven, D. M. W. Leenaerts, Explicit formulas for the solutions of piecewise linear networks, *IEEE trans. Circuits Systems I Fund. Theory Appl.*, Vol.46, No.9, pp.1110–1117, (1999).

[2] R.W. Cottle, J.-S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, (1992).

[3] J. T. J. Eijndhoven, Piecewise linear analysis, in T. Ozawa (ed.), *Analog methods for computer-aided circuit analysis and diagnosis 1*, Marcel Dekker, New York, (1988).

[4] V.V. Gorokhovik and O.I. Zorko, Piecewise affine functions and polyhedral sets, *Optimization*, **31** (1994) 209–221.

[5] W. P. M. H. Heemels, B. D. Schutter, A. Bemporad, On the equivalence of classes of hybrid dynamical models, *Proc. of the 40th IEEE Conference on Decision and Contol*, Orlando, Florida, pp.364–369, Dec. (2001).

[6] M. Johansson, *Piecewise Linear Control Systems: A computational approach*, Lecture Notes in Control and Information Sciences No.284, Springer, (2003).

[7] T. A. Kevenaar, D. M. W. Leenaerts, W. M. G. van Bokhoven, Extensions to Chua's explicit piecewise-linear function descriptions, *IEEE trans. Circuits Systems I Fund. Theory Appl.*, Vol.41, No.4, pp.308–314, (1994).

[8] D.M.W. Leenaerts and W.M.G. van Bokhoven, *Piecewise Linear Modeling and Analysis*, Kluwer, Boston (1998).

[9] S. Ovchinnikov, Max-min representation of piecewise linear functions, *Contributions to Algebra and Geometry*, **43** (2002) 297–302.

[10] T. Sakurai, T. Murofushi, Representations of piecewise linear functions and Choquet integral over a finite set, *8th Workshop on Evaluation of Heart and Mind*, pp.55–63, (2003) (in Japanese).

[11] T. Sakurai, T. Murofushi, The P- and ULT-representations of piecewise linear functions, *23th Fuzzy Workshop / 9th Workshop on Evaluation of Heart and Mind*, pp.19–22, (2004) (in Japanese).